An investigation of Kepler's first law of planetary motion

Introduction:

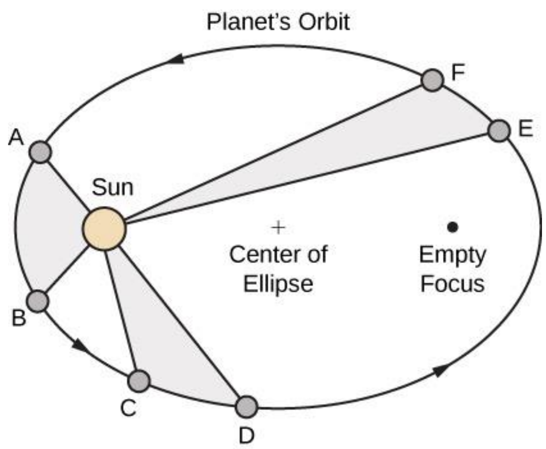
In this investigation, my aim is to find a link between the eccentricity of the planets and their orbital shapes, which is based on the three laws created by Johannes Kepler in the 17th century.

I chose this topic because I have always liked the field of astrophysics and particularly how celestial bodies such as planets and asteroids move with respect to the stars that they orbit. And when I read that Kepler created three laws of planetary motion, I knew that this could be a good topic for a math investigation.

Background information:

- Between 1609 and 1619, Johannes Kepler published three laws of planetary motion with the help of data collected by his mentor Tycho Brahe.
- In these laws, Kepler improved upon the heliocentric model of the solar system put forth by Copernicus by saying that instead of circular orbits, planets in the solar system actually revolved around the sun in elliptical orbits, in which their velocities varied depending on their position in the orbit with respect to their distance from the sun.
- The three laws Kepler put forth are:
    - The law of ellipses: The path of any planet about the Sun is elliptical in shape, with the centre of the Sun located at one focus of the ellipse
    - The law of equal areas: A line drawn from the centre of the Sun to the centre of a planet sweeps out equal areas in equal time intervals

- The law of harmonies: The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes



- Above is the diagrammatic representation of the first and second laws. This investigation is going to find the relationship between the position of the sun (a focus / centre of the eclipse / or in between) and its effect on the shape of planetary orbits.

- In this investigation, multiple theories and problems will be used to first prove Kepler's first law of planetary motion, and then investigate the effect of eccentricity in that particular model.

Newton's laws of universal gravitation

Even though the laws of planetary motion were formulated by Johannes Kepler, it was Sir Isaac Newton who gave mathematical proof that the laws were true. He did this with the help of his law of universal gravitation which has the following properties:

$$F \propto M \cdot m$$

$$F \propto \frac{1}{r^2}$$

The first property says that the gravitational force(F), is directly proportional to the product of the two masses in question, which is the sun and a planet in the case of the solar system (M and m). The second property says that the gravitational force is inversely proportional to the distance between the two objects. These two properties combine to give us the formula for gravitational force:

$$F = G \cdot \frac{M \cdot m}{r^2} \qquad [1]$$

Where G is the gravitational constant given by Isaac Newton which has a value of $6.67 \cdot 10^{-11}$. When this formula is used with the proper substituted values, the gravitational force between the two objects is obtained.

Another formula given by Newton is:

$$F = m \cdot a \qquad [2]$$

Where “m” is the mass of the object and “a” is the acceleration of the object. This formula comes from Newton’s second law of motion. And when you combine the equations 1 and 2, we get a new equation for acceleration:

$$m \cdot a = G \cdot \frac{M \cdot m}{r^2}$$

$$\boldsymbol{a = G \cdot \frac{M}{r^2}} \qquad [3]$$

Now that we know the formula for gravitational force, we can apply it to a classical problem called the two body problem.

The two-body problem

The two-body problem describes how two gravitating objects move as a function of time. This problem uses vectors, differential calculus and Newton’s law of gravitation to create a relationship between the two objects.

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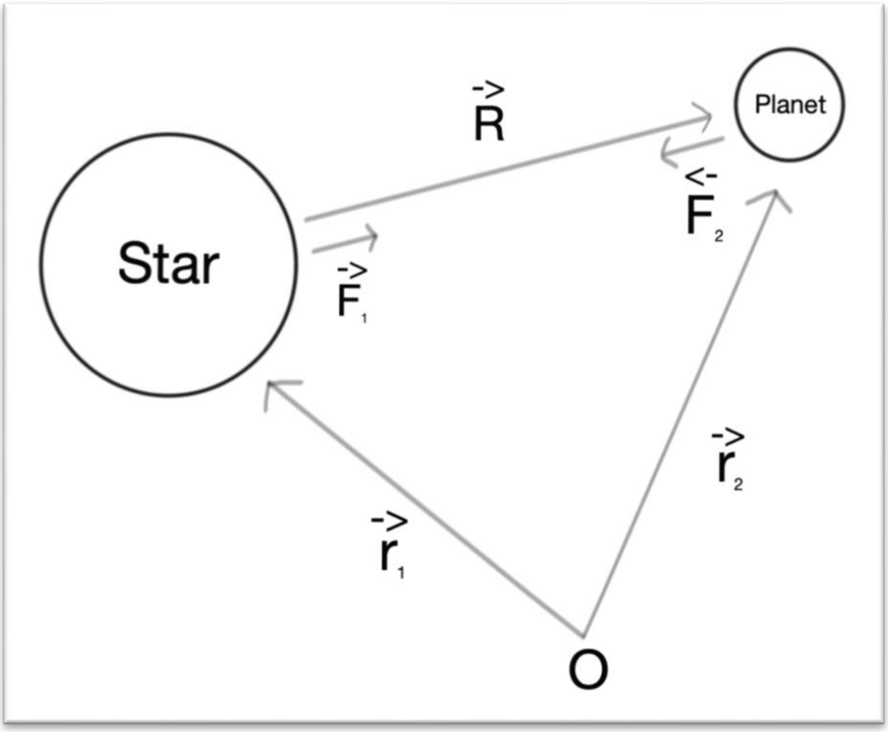
Before solving the problem, we must visualize it.



In the image above, the vectors r_1 and r_2 are shown to be going from the reference origin O to the star and the planet. The star has a mass of “M” and the planet has a mass of “m”. The vectors R, r_1, and r_2 are position vectors. The vectors f_1 and f_2 show the gravitational force going from the star to the planet, and following Newton’s third law of motion, from the planet to the star. Now we can solve the two body problem.

Consider that:

$$\vec{r} = r \cdot \hat{r}$$

The above formula says that the vector r is the combination of the scalar quantity r and the unit vector r. this will be useful later.

Now let us go back to the diagram. From the diagram and Newton’s law of universal gravitation that we saw earlier, we can see that the gravitational force in this situation will be:

$$F = G \cdot \frac{M \cdot m}{r^2}$$

But since now we are dealing in vectors, we must make a small change to it:

$$\vec{F} = G \cdot \frac{M \cdot m}{r^2} \cdot \hat{r} \qquad [4]$$

By multiplying the unit vector, we have made the quantity “F” a vector. However in the diagram, the “F” quantities are in opposite directions, and if we consider them to be equal in magnitude and apply Newton’s second law of motion, we get:

$$\vec{F}_1 = M \cdot \vec{a}_1 \qquad [5]$$

$$\vec{F}_2 = m \cdot \vec{a}_2 \qquad [6]$$

These equations will help us find the acceleration of these bodies caused by their gravitational fields.

From kinematics, it is known that acceleration is the derivative of velocity with respect to time, and velocity is the derivative of displacement with respect to time. So in this case, it becomes:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Putting this back into equations 5 and 6 we get:

$$\vec{F}_1 = M \cdot \frac{d^2\overrightarrow{r_1}}{dt^2} \qquad [7]$$

$$\vec{F}_2 = m \cdot \frac{d^2\overrightarrow{r_2}}{dt^2} \qquad [8]$$

And when we combine the above equations with equation 4 and the second law of motion, we get:

$$\vec{F}_1 \rightarrow M \cdot \frac{d^2\overrightarrow{r_1}}{dt^2} = G \cdot \frac{M \cdot m}{r^2} \cdot \hat{r}$$

$$\vec{F}_1 \rightarrow \frac{d^2\overrightarrow{r_1}}{dt^2} = G \cdot \frac{m}{r^2} \cdot \hat{r} \qquad [9]$$

$$\vec{F}_2 \rightarrow m \cdot \frac{d^2\overrightarrow{r_2}}{dt^2} = -G \cdot \frac{M \cdot m}{r^2} \cdot \hat{r}$$

$$\vec{F}_2 \rightarrow \frac{d^2\overrightarrow{r_2}}{dt^2} = -G \cdot \frac{M}{r^2} \cdot \hat{r} \qquad [10]$$

Note that "M" and "m" have been cancelled in equations 9 and 10 respectively. Also note that the RHS of equation 10 has a negative sign to it. This is because it is in the opposite direction of F_1.

Now, the final step of the two body problem is to combine equations 9 and 10. For this, we use a vector property called resolution of vectors:

$$\vec{r} = \overrightarrow{r_1} + \overrightarrow{r_2}$$

This property tells us that the combination of vectors r_1 and r_2 gives us the vector r. When we derive the equation twice, we get:

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\overrightarrow{r_1}}{dt^2} + \frac{d^2\overrightarrow{r_2}}{dt^2} \qquad [11]$$

Now we subtract equation 9 from equation 10 using equation 11:

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\overrightarrow{r_1}}{dt^2} + \frac{d^2\overrightarrow{r_2}}{dt^2}$$

$$\frac{d^2\vec{r}}{dt^2} = -G \cdot \frac{M}{r^2} \cdot \hat{r} - G \cdot \frac{m}{r^2} \cdot \hat{r}$$

$$\frac{d^2\vec{r}}{dt^2} + G \cdot \frac{(M+m)}{r^2} \cdot \hat{r} = 0 \qquad [12]$$

Note that after subtraction, the RHS was brought to the LHS and became positive while the RHS got the value of 0.

Now, the equation 12 must be solved to find the time evolution of the distance vector (r) between the planet and the star. For this we will use a method called the cross product of vectors.

The cross product of vectors

To understand how the equation 12 describes a planet's motion around a star, we use the cross product of vectors.

If there are vectors A and B, the cross product of these vectors would be:

$$\vec{A} \times \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \sin\theta \cdot \hat{n}$$

In the above equation, the magnitude of vectors A and B is multiplied by the sine of the angle created between them and the unit vector n, which is perpendicular to vectors A and B.

Now we cross the vector r with equation 12:

$$\vec{r} \times \left(\frac{d^2\vec{r}}{dt^2} + G \cdot \frac{(M+m)}{r^2} \cdot \hat{r} = 0\right)$$

$$\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) + \left(G \cdot \frac{(M+m)}{r^2} \cdot \vec{r} \times \hat{r}\right) = 0$$

Since the cross product of the vector r and the unit vector r is 0 (the angle between them is 0, and the sine of 0 is 0), we get:

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0 \qquad [13]$$

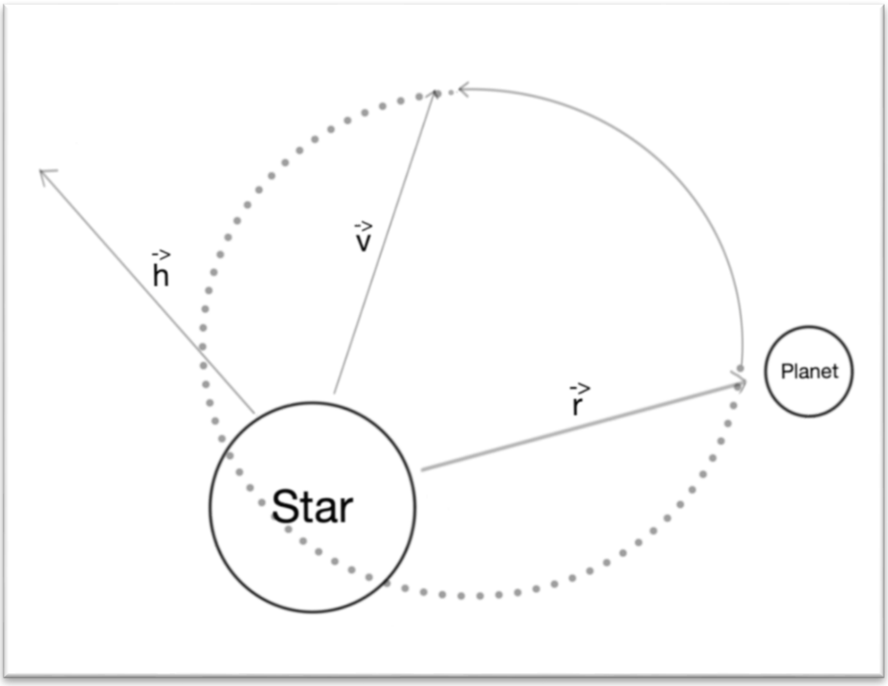
Note that:

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right)$$

Therefore we can conclude that:

$$\vec{r} \times \frac{d\vec{r}}{dt} = \vec{h} \qquad [14]$$

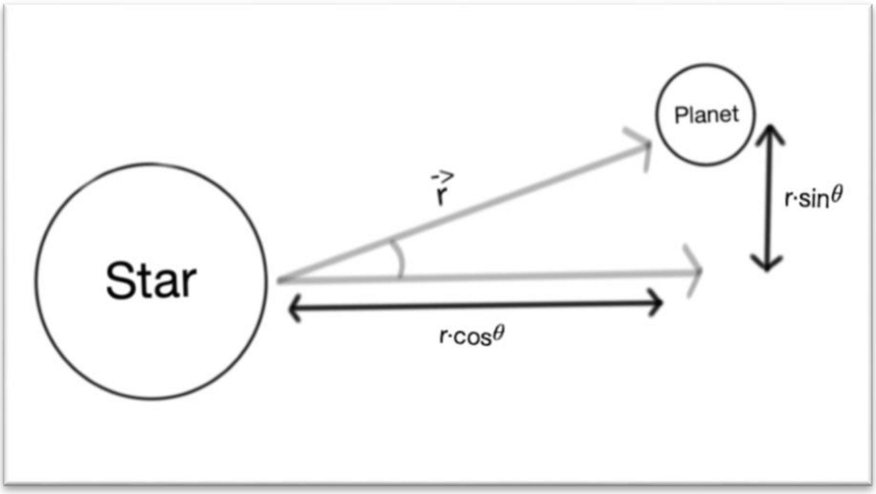
In the above equation, the vector h is a constant vector obtained from the RHS. The vector h is the angular momentum which we will use later. When equation 14 is visualised, we get:



In the picture above, the vectors v and r can be seen perpendicular to the vector h, which is the constant vector from the equation above. The curved line in the image is the path of the planet. The dotted line represents the plane created by the creation of vector h, in which the planet travels. Therefore at this point we know that the planet only travels in one plane. Now that it has become a 2-dimentional space instead of a 3-dimentional one, we can use the plane polar co-ordinate system to further solve the problem.

The plane polar co-ordinate system

Now that we can imagine the situation in 2D, we can use polar co-ordinates and trigonometry:



The diagram above says that the distance in the horizontal direction (x-axis) is r·cosθ, and the distance in the vertical direction is r·sinθ. This is obtained from using polar co-ordinates and trigonometry.

Now the vector r can be defined as a column vector:

$$\vec{r} = \binom{r \cdot cos\theta}{r \cdot sin\theta} \qquad [15]$$

This can be simplified to:

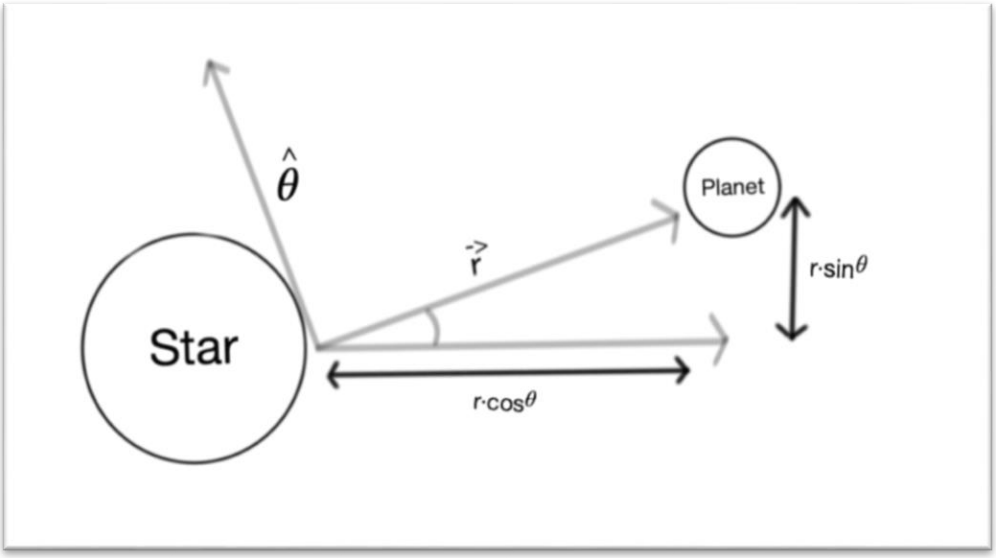
$$\vec{r} = r \cdot \binom{cos\theta}{sin\theta}$$

Now, since

$$\vec{r} = r \times \hat{r}$$

$$\hat{r} = \binom{cos\theta}{sin\theta} \qquad [16]$$

Following the angular momentum model, we can define another constant vector in the plane polar co-ordinate system which is perpendicular to the vectors present:



The constant unit vector θ can be defined as:

$$\hat{\theta} = \binom{-sin\theta}{cos\theta} \qquad [17]$$

The above can be figured out by the same method used for the vector r. Now that we have all the components from our plane polar co-ordinate system, we can calculate the vector h.

Formulating the equation for distance between the planet and the star

To formulate the equation for distance, all the equations created up till now must be combined:

$$\vec{r} = r \cdot \hat{r}$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(r \cdot \hat{r})$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} \cdot \hat{r} + \frac{d\hat{r}}{dt} \cdot r$$

Using equation 16:

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} \cdot \hat{r} + \frac{d}{dt}\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \cdot r$$

Using the chain rule:

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} \cdot \hat{r} + \frac{d\theta}{dt} \cdot \frac{d}{d\theta}\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \cdot r$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} \cdot \hat{r} + \frac{d\theta}{dt} \cdot \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \cdot r$$

Using equation 17:

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} \cdot \hat{r} + \frac{d\theta}{dt} \cdot \hat{\theta} \cdot r$$

Now we can use this equation and substitute it in equation 14:

$$\vec{h} = \vec{r} \times \frac{d\vec{r}}{dt}$$

$$\vec{h} = \vec{r} \times \left(\frac{dr}{dt} \cdot \hat{r} + \frac{d\theta}{dt} \cdot \hat{\theta} \cdot r\right)$$

$$\vec{h} = \left(\vec{r} \times \hat{r} \cdot \frac{dr}{dt}\right) + \left(\vec{r} \times \hat{\theta} \cdot \frac{d\theta}{dt} \cdot r\right)$$

$$\vec{h} = \vec{r} \times \hat{\theta} \cdot \frac{d\theta}{dt} \cdot r \qquad [18]$$

Note that the first bracket on the RHS becomes 0 since the cross product of the vector r and the unit vector r is 0.

From equation 17, we can make:

$$\vec{h} = \hat{r} \times \hat{\theta} \cdot \frac{d\theta}{dt} \cdot r^2 \qquad [19]$$

Note that the vector r was resolved to get the unit vector r and the scalar quantity r. The scalar quantity was multiplied to the scalar quantity already present, therefore resulting in equation 19.

In equation 19, the cross product of unit vectors r and θ will be perpendicular to the rest of the vectors in the model. This cross product will be the unit vector called h. Therefore we have another equation:

$$\vec{h} = r^2 \cdot \frac{d\theta}{dt} \cdot \hat{h} \qquad [20]$$

From equation 20 and the principle of the resolution of vectors, it can be formulated that:

$$\vec{h} = h \cdot \hat{h}$$

$$h = r^2 \cdot \frac{d\theta}{dt} \qquad [21]$$

Now that we have the value for the scalar quantity h, we can insert it in earlier equations and formulate an equation for angular momentum:

$$\frac{d^2\vec{r}}{dt^2} + G \cdot \frac{(M+m)}{r^2} \cdot \hat{r} = 0$$

In the above equation, the value of acceleration is still unknown, for that the velocity equation can be derived:

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt} \cdot \hat{r} + \frac{d\theta}{dt} \cdot \hat{\theta} \cdot r$$

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} \cdot \hat{r} + \frac{dr}{dt} \cdot \frac{d\hat{r}}{dt}\right) + \left(\frac{dr}{dt} \cdot \frac{d\theta}{dt} \cdot \hat{\theta} + \frac{d^2\theta}{dt^2} \cdot r \cdot \hat{\theta} + \frac{d\theta}{dt} \cdot \frac{d\hat{\theta}}{dt} \cdot r\right) \qquad [22]$$

Note that the product rule has been used for differentiation of two terms and three terms. In the above equation:

$$\frac{d\hat{\theta}}{dt} = \frac{d\theta}{dt} \cdot \frac{d}{d\theta}\begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d\theta}{dt} \cdot -\hat{r} \qquad [23]$$

This is because of equation 16.

Now, equation 22 becomes:

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2}\cdot\hat{r} + \frac{dr}{dt}\cdot\frac{d\hat{r}}{dt}\right) + \left(\frac{dr}{dt}\cdot\frac{d\theta}{dt}\cdot\hat{\theta} + \frac{d^2\theta}{dt^2}\cdot r\cdot\hat{\theta} + \frac{d\theta}{dt}\cdot -\hat{r}\cdot r\right)$$

After opening the brackets we get:

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2r}{dt^2}\cdot\hat{r} + 2\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt}\cdot\hat{\theta} + r\cdot\frac{d^2\theta}{dt^2}\cdot\hat{\theta} - r\cdot\left(\frac{d\theta}{dt}\right)^2\cdot\hat{r}$$

$$\frac{d^2\vec{r}}{dt^2} = \hat{r}\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + \hat{\theta}\left(2\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\cdot\frac{d^2\theta}{dt^2}\right) \qquad [24]$$

Now we substitute equation 24 into equation 12:

$$\hat{r}\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + G\cdot\frac{(M+m)}{r^2}\cdot\hat{r} = 0 \qquad [25]$$

Note that only half of the equation 24 was substituted. This is because the theta part of the equation is used for other investigations of planetary motion and does not contribute to shapes of the orbit anymore.

$$\hat{r}\left(\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) + G\cdot\frac{(M+m)}{r^2}\right) = 0 \qquad [26]$$

And since:

$$h = r^2\cdot\frac{d\theta}{dt}$$

$$\frac{h}{r^2} = \frac{d\theta}{dt}$$

Equation 26 becomes:

$$\hat{r}\left(\left(\frac{d^2r}{dt^2} - r\cdot\frac{h}{r^4}\right) + G\cdot\frac{(M+m)}{r^2}\right) = 0$$

$$\hat{r}\left(\left(\frac{d^2r}{dt^2} - \frac{h}{r^3}\right) + G\cdot\frac{(M+m)}{r^2}\right) = 0$$

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Now inside the bracket, there are no vectors, and that is what we need to calculate the orbital shape. So:

$$\left(\frac{d^2r}{dt^2} - \frac{h}{r^3}\right) + G \cdot \frac{(M+m)}{r^2} = 0 \qquad [27]$$

The equation 27 cannot be solved as a function of time, so a unit transformation has to be made using substitution differentiation so that it can be solved in terms of r with respect to θ.

Let:

$$u = \frac{1}{r} \qquad [28]$$

$$r = \frac{1}{u} \qquad [29]$$

Using the chain rule:

$$\frac{dr}{dt} = \frac{dr}{du} \cdot \frac{du}{dt} \qquad [30]$$

Using equation 29, in equation 30:

$$\frac{dr}{du} = \frac{-1}{u^2} \qquad [31]$$

Therefore equation 30 becomes:

$$\frac{dr}{dt} = -r^2 \cdot \frac{du}{dt}$$

Using chain rule:

$$\frac{dr}{dt} = -r^2 \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt}$$

Using equation 21:

$$\frac{dr}{dt} = -r^2 \cdot \frac{du}{d\theta} \cdot \frac{h}{r^2}$$

$$\frac{dr}{dt} = \frac{du}{d\theta} \cdot -h \qquad [32]$$

Differentiating equation 32:

$$\frac{d^2r}{dt^2} = \left(\frac{d}{dt}\frac{du}{d\theta} \cdot -h\right) + \left(\frac{du}{d\theta} \cdot \frac{dh}{dt}\right)$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\frac{du}{d\theta} \cdot -h \qquad [33]$$

Using chain rule in equation 33:

$$\frac{d^2r}{dt^2} = \frac{d\theta}{dt} \cdot \frac{d^2u}{d\theta^2} \cdot -h$$

Using equation 21:

$$\frac{d^2r}{dt^2} = -h^2 \cdot \frac{1}{r^2} \cdot \frac{d^2u}{d\theta^2}$$

Using equation 28:

$$\frac{d^2r}{dt^2} = -h^2 \cdot u^2 \cdot \frac{d^2u}{d\theta^2} \qquad [34]$$

Now equation 28 can be substituted in equation 27:

$$\left(\frac{d^2r}{dt^2} - \frac{h}{r^3}\right) + G \cdot \frac{(M+m)}{r^2} = 0$$

$$\left(-h^2 \cdot u^2 \cdot \frac{d^2u}{d\theta^2} - \frac{h}{r^3}\right) + G \cdot \frac{(M+m)}{r^2} = 0$$

$$-h^2 \cdot u^2 \cdot \frac{d^2u}{d\theta^2} - \frac{h}{r^3} = -G \cdot \frac{(M+m)}{r^2}$$

$$\frac{d^2u}{d\theta^2} + u = G \cdot \frac{(M+m)}{h^2} \qquad [35]$$

Now we can calculate u in terms of θ, and then use equation 29 to find it in terms of r, which was our primary objective for doing the unit transformation.

Now, a function u is required to be substituted in equation 35. This function must give out a constant when differentiated twice and added to itself. This is because the RHS of the equation 35 is a constant. For example:

$$u(\theta) = A \cdot \sin\theta + k \qquad [36]$$

$$\frac{d^2u}{d\theta^2} = A \cdot -\sin\theta$$

$$u(\theta) + \frac{d^2u}{d\theta^2} = constant$$

Or:

$$u(\theta) = A \cdot \cos\theta + k \qquad [37]$$

$$\frac{d^2u}{d\theta^2} = A \cdot -\cos\theta$$

$$u(\theta) + \frac{d^2u}{d\theta^2} = constant$$

Combining equations 36 and 37, u can be defined as:

$$u(\theta) = k \cdot \cos(\theta - \varphi) + G \cdot \frac{(M+m)}{h^2} \qquad [38]$$

In equation 38, k is the constant (A in equations 36 and 37) and phi is the reference angle. This value of the function u can be verified if it is differentiated twice and substituted in equation 35.

Equation 38 can be written as:

$$u(\theta) = G \cdot \frac{(M+m)}{h^2}(1 + e \cdot \cos\theta - \varphi) \quad [39]$$

Here, e is a constant which has a value of:

$$e = \frac{k}{G \cdot \frac{(M+m)}{h^2}}$$

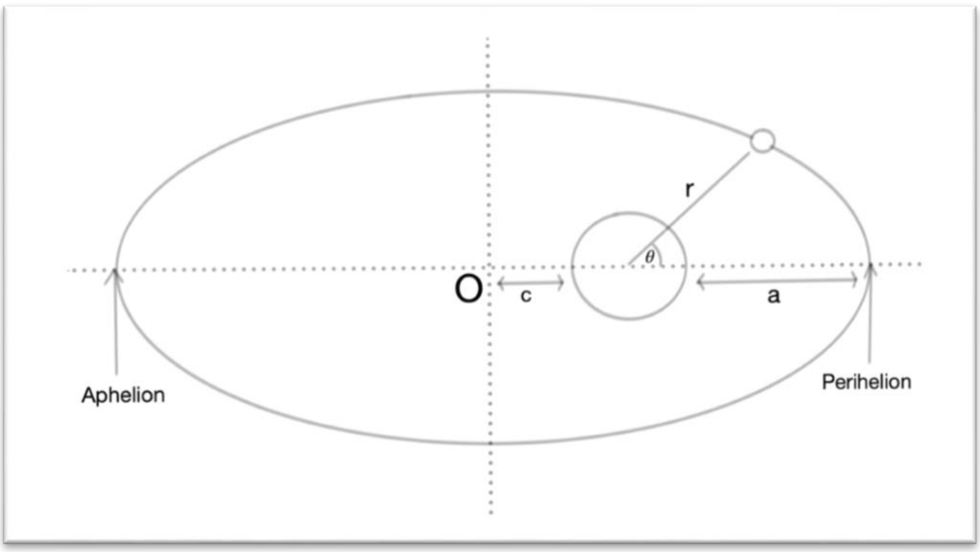
Now that we have a definite value of u, we can define r using the equation 29:

$$r(\theta) = \frac{h^2}{G(M+m)}\left(\frac{1}{1+e\cdot\cos\theta}\right) \quad [40]$$

Note that the value of phi has been considered to be 0.

Now we have a value of the function r with respect to its angle, which tells us how the distance between a star and a planet varies as a function of the angle, which is basically the shape of the orbit. Still, we need to calculate the value of h, since it is unknown.

Orbit of a planet



In the diagram above,

- O= the centre of the orbit
- c= the distance between the star and the centre of the orbit
- a= the distance between the star and the orbit
- r= the distance between the star and the planet
- θ= the angle between the star and the planet with respect to the centre
- Perihelion: the point in a planet's orbit where it is closest to the star
- Aphelion: the point in a planet's orbit where it is farthest from the star

Eccentricity of an orbit is:

$$e = \frac{c}{a}$$

Perihelion can be defined as:

$$a - c$$

$$a - ae$$

$$a(1 - e) \qquad [41]$$

Aphelion can be defined as:

$$a + c$$

$$a + ae$$

$$a(1 + e) \qquad [42]$$

Value of h

Now, the value of perihelion can be used to further dissociate equation 40 and find the value of h:

$$r = \frac{h^2}{G(M+m)}\left(\frac{1}{1+e\cdot\cos\theta}\right)$$

$$a(1-e) = \frac{h^2}{G(M+m)}\left(\frac{1}{1+e}\right)$$

Note that the cos function disappeared. This is because at the perihelion, θ becomes 0, and cos of 0 is 1.

$$h^2 = a(1-e)^2 \cdot G(M+m)$$

The distance between a planet and a star

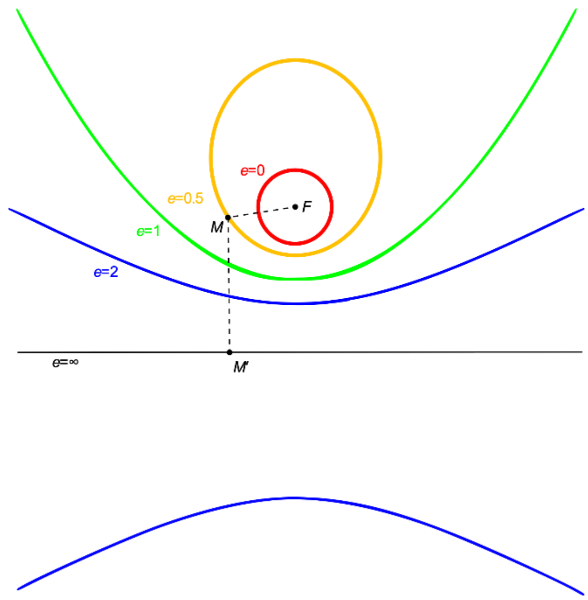
Now that we have an expression for h^2, we can substitute it in equation 40:

$$r = \frac{a(1-e)^2 \cdot G(M+m)}{G(M+m)}\left(\frac{1}{1+e\cdot\cos\theta}\right)$$

$$r = \frac{a(1-e)^2}{1+e\cdot\cos\theta} \qquad [43]$$

Visualising equation 43

When the appropriate values are substituted in the equation and the graph is plotted, the shape of the orbit depends on the "e" value in the equation which represents the eccentricity of the orbit.



The image above shows the orbital shapes with respect to the eccentricity values.

Conclusion

After all the calculations, the relationship between eccentricity and the shape of the orbit has been determined. As the value of eccentricity increases, the shape of the orbit goes more and more away from the circular orbit, as is shown in the table below.

Eccentricity value	Shape of orbit
0	Circle
0 < e < 1	Ellipse
1	Parabola
e > 1	Hyperbola

Therefore, the relationship between the eccentricity values and the shape of orbits has been determined, answering the research question with conclusive evidence.

If this investigation were to be extended, the relationship between the eccentricity value and the acceleration of planets around the orbits can be explored. This is a viable further investigation because as the shape of the orbit becomes more elliptical, the acceleration of the planet changes throughout the path of its orbit whereas if it is a circular orbit, the acceleration of the planet will be uniform throughout the period of revolution.

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